

Math 60 7.7 Solving Rational Equations

Objectives:

- 1) Recognize a Rational Equation
 - a. Has an = sign
 - b. Instructions say “solve” (not “simplify”, 7.1-7.6)
 - c. One or more terms is a fraction
 - d. The denominator of one or more fractions contains a variable.
 - e. Method for solving is Clear the Fractions.
- 2) Solve equations containing rational expressions for value(s) of the variable.
 - a. Clear fractions from the equation.
 - i. Factor all denominators and find the LCD.
 - ii. Add parentheses around any numerator with two or more terms, or a negative term.
 - iii. Multiply all terms by LCD, including those that have no denominator.
 - iv. Cancel all denominators.
 - v. Multiply all numerators using distribute or FOIL if necessary.
 - vi. Solve the remaining equation using an appropriate method.
 1. Linear (degree 1) equations – isolate the variable
 2. Quadratic or Cubic (degree 2 or 3 equations) – factor, set factors equal to 0, solve.
 - a. After chapter 10, also may use Quadratic Formula
 - b. Identify values of the variable that cause any part of the equation to be undefined.
 - i. Eliminate extraneous solutions which cause undefined in the original equation
- 3) Isolate a specified variable in an equation containing rational expressions. (Solve for a variable.)
 - a. Clear fractions using method above
 - b. Solve for the specified variable using techniques from Math 45.

Warm-up and Review:

- 1) Solve $8k - 2(k+1) = 6$
- 2) Solve $8z^2 - 10z - 3 = 0$
- 3) Find the values of the variable that make the expression undefined: $\frac{x+4}{x^2-2x-24}$
- 4) Solve $4x - 2y = 10$ for y
- 5) Solve $\frac{3x+2}{4} - \frac{x}{12} = \frac{x}{3} - 1$

Examples and Practice:

Solve.

6) $\frac{2}{p} - \frac{4}{3p} = -\frac{1}{6}$

7) $\frac{3}{x-5} = -\frac{4}{x+2}$

$$8) \frac{2}{w} - \frac{10}{w-3} = 1$$

$$9) \frac{2}{x+1} - \frac{5}{x-2} = \frac{x^2 - 6x - 13}{x^2 - x - 2}$$

$$10) \frac{-12}{x^2 - 9} = \frac{2}{x+3} - \frac{5}{x-3}$$

Solve for the specified variable.

$$11) \frac{a}{1+b} = c \text{ for } b$$

$$12) \frac{1}{x} = \frac{1}{y} + \frac{1}{z} \text{ for } y$$

$$13) x = \frac{ab}{a+b} \text{ for } a$$

The average daily cost $\bar{C}$, in dollars, of manufacturing x bicycles is given by the equation $\bar{C} = \frac{x^2 + 75x + 5000}{x}$

14) Determine the number of bicycles for which the average daily cost will be \$240.

Solve.

$$15) \frac{4}{x-3} - \frac{3}{x-2} = \frac{2x+1}{x^2 - 5x + 6}$$

$$16) \frac{6}{t} - \frac{2}{t-1} = \frac{2-4t}{t^2 - t}$$

$$17) \frac{5}{x+1} - \frac{7}{2} = \frac{-5x-9}{2x+2}$$

Simplify.

$$18) 5 - \frac{n^2 - 3n - 4}{n^2 - 4}$$

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① Solve $8k - 2(k+1) = 6$

$$8k - 2k - 2 = 6$$

$$6k - 2 = 6$$

$$6k = 8$$

$$k = \frac{8}{6}$$

$$\boxed{k = \frac{4}{3}}$$

dist -2

combine like terms

add 2 to both sides to isolate k.

divide both sides by 6

reduce fraction to lowest terms

② Solve $8z^2 - 10z - 3 = 0$

$$(2z-3)(4z+1) = 0$$

factor $8z^2 - 10z - 3$

$$\begin{array}{r} -24 \\ -12 \times +2 \\ -10 \end{array}$$

$$8z^2 - 12z + 2z - 3$$

$$4z(2z-3) + 1(2z-3)$$

$$(2z-3)(4z+1)$$

$$2z - 3 = 0 \quad 4z + 1 = 0$$

$$2z = 3 \quad 4z = -1$$

$$\boxed{z = \frac{3}{2} \quad z = -\frac{1}{4}}$$

or $\boxed{\left\{ \frac{3}{2}, -\frac{1}{4} \right\}}$

Set each factor equal to 0
(using zero-product property)

Isolate variable in each
subordinate equation.

Either form of answer
is OK.

MathXL will expect a
solution set (usually).

③ Find values that make $\frac{x+4}{x^2-2x-24}$ undefined

$$x^2 - 2x - 24 = 0$$

Undefined occurs when we
get $\div 0$, so only denominator
must be zero.

$$(x-6)(x+4) = 0$$

Factor $x^2 - 2x - 24$

$$\begin{array}{r} -24 \\ -6 \times 4 \\ -2 \end{array}$$

$$x - 6 = 0 \quad x + 4 = 0$$

Set factors = 0.

$$\boxed{x = 6 \quad x = -4}$$

Isolate variable

or $\boxed{\{6, -4\}}$

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④ Solve $4x - 2y = 10$ for y .

$$\begin{array}{r} 4x - 2y = 10 \\ \underline{-4x} \quad \underline{-4x} \end{array}$$

$$\begin{array}{r} -2y = -4x + 10 \\ \underline{-2} \quad \underline{-2} \quad \underline{-2} \end{array}$$

y isolated $\rightarrow$ $y = 2x - 5$

subtract $4x$ from both sides

divide all terms by -2

reduce all fractions

⑤ Solve $\frac{3x+2}{4} - \frac{x}{12} = \frac{x}{3} - 1$

Find LCD of 4, 12, 3 (all denominators)

LCD = 12

$$12 \left(\frac{3x+2}{4} \right) - \frac{12 \cdot x}{12} = \frac{12 \cdot x}{3} - 12 \cdot 1$$

Mult all terms by LCD

*even when there is no denominator

Add parentheses around any numerator with two or more terms

reduce fractions

$$\frac{12}{4}, \frac{12}{12}, \frac{12}{3}$$

$$3(3x+2) - x = 4x - 12$$

$$9x + 6 - x = 4x - 12$$

$$8x + 6 = 4x - 12$$

$$4x = -18$$

$$x = \frac{-18}{4}$$

$$x = \frac{-9}{2}$$

dist $3(3x+2)$

combine like terms

subtract $4x$ } both sides
subtract 6 }

divide 4

reduce

⑥ $\frac{2}{p} - \frac{4}{3p} = -\frac{1}{6}$

step 1: Find LCD of all denoms

$p, 3p, 6 \Rightarrow 6p = \text{LCD}$

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$$\frac{6p \cdot 2}{1 \cdot p} - \frac{6p \cdot 4}{1 \cdot 3p} = \frac{6p \cdot (-1)}{1 \cdot 6}$$

Mult all terms by $\frac{LCD}{1}$

Add () around negative numerator

$$6 \cdot 2 - 2 \cdot 4 = p(-1)$$

Cancel denominators
Write remaining expression

$$12 - 8 = -p$$

$$4 = -p$$

$$-4 = p$$

Simplify each term

Combine like terms

Isolate p.

*** CAUTION * IMPORTANT * CAUTION * IMPORTANT ***

Although multiplying by the LCD is the only correct method for solving a rational equation, it can create solutions that do not work, because they cause divide by 0 ($\frac{\quad}{0}$) when you plug them in. These wrong solutions are called extraneous solutions and we cross them off (reject them) and do not include them in the final boxed answer.

YOU MUST VERIFY EVERY SOLUTION DOES NOT CAUSE $\div 0$.

Two methods to verify:

Method 1: Plug in and confirm it works

*** Must use original equation with fractions ***

$$\frac{2}{(-4)} - \frac{4}{3(-4)} = \frac{-1}{6}$$

$$\frac{-2}{4} + \frac{4}{12} = \frac{-1}{6}$$

$$-\frac{1}{2} + \frac{1}{3} = \frac{-1}{6} \checkmark$$

yes, you can use your calculator

Method 2: Determine values of variable that make expressions in the equation undefined. If the answer is one of these values, reject it.

$$\frac{2}{p} \text{ is undefined if } p=0.$$

$$\frac{4}{3p} \text{ is undefined if } 3p=0 \Rightarrow p=0$$

$$-\frac{1}{6} \text{ is never undefined.}$$

The only excluded value is 0, but we did not get 0, so our answer is valid and not extraneous.

$$\boxed{p = -4} \text{ or } \boxed{\frac{2}{3} - 4\frac{2}{3}}$$

$$\textcircled{7} \quad \frac{3}{x-5} = -\frac{4}{x+2}$$

$$\frac{3}{x-5} = \frac{-4}{x+2}$$

* CAUTION *

Don't lose the negative on the RHS.

Suggest moving it to the numerator.

$$\frac{3(\cancel{x-5})(x+2)}{(\cancel{x-5})} = \frac{-4(\cancel{x-5})(\cancel{x+2})}{(\cancel{x+2})} \quad \text{LCD} = (x-5)(x+2)$$

mult all terms by LCD.
add ()

cancel denominators

$$3(x+2) = -4(x-5)$$

NOTE: Because this is
(one fraction) = (one fraction)

We could have cross-multiplied to get directly to this step.

$$3x + 6 = -4x + 20$$

$$7x = 14$$

$$x = 2$$

$$\text{dist } 3(x+2)$$

$$\text{dist } -4(x-5)$$

isolate x

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Verify
Method 1:

$$\frac{3}{2-5} = \frac{-4}{2+2}$$

$$\frac{3}{-3} = \frac{-4}{4} \checkmark$$

$$\boxed{x=2} \quad \text{or} \quad \boxed{\{2\}}$$

Method 2:

$$\frac{3}{x-5} \text{ undefined } x=5$$

$$\frac{-4}{x+2} \text{ undefined } x=-2$$

+2 is neither of these.

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$$\textcircled{8} \quad \frac{2}{w} - \frac{10}{w-3} = 1$$

$$\text{LCD} = w(w-3)$$

$$\frac{2}{\cancel{w}} \cdot \cancel{w}(w-3) - \frac{10}{\cancel{(w-3)}} \cdot w(\cancel{w-3}) = 1 \cdot w(w-3)$$

Mult all terms by LCD
* including RHS

$$2(w-3) - 10w = w(w-3)$$

$$2w - 6 - 10w = w^2 - 3w$$

$$-8w - 6 = w^2 - 3w$$

cancel denoms
rewrite remaining terms

dist $2(w-3)$

dist $w(w-3)$

combine like terms

$$0 = w^2 + 5w + 6$$

Notice w^2 makes this quadratic
Set = 0, factor
(not isolate variable)

$$\begin{array}{r} 6 \\ 2 \times 3 \\ 5 \end{array}$$

$$0 = (w+2)(w+3)$$

$$w+2=0 \quad w+3=0$$

$$w=-2 \quad w=-3$$

add $8w$ and 6 to both sides to set = 0.

factor, set factors = 0.

Verify
 $w=-2$:

$$\frac{2}{-2} - \frac{10}{-2-3} = 1$$

$$-1 - \frac{10}{-5} = 1$$

$$-1 + 2 = 1 \quad \checkmark$$

Verify
 $w=-3$:

$$\frac{2}{-3} - \frac{10}{-3-3} = 1$$

$$-\frac{2}{3} - \frac{10}{-6} = 1$$

$$-\frac{2}{3} + \frac{5}{3} = 1 \quad \checkmark$$

Must verify both!

$$\frac{2}{w} \text{ undefined } w=0$$

$$\frac{10}{w-3} \text{ undefined } w=3$$

1 always defined.

$w=-2$ and $w=-3$ are not these values.

$$\boxed{w=-2 \quad w=-3}$$

or

$$\boxed{\{-2, -3\}}$$

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$$(9) \quad \frac{2}{x+1} - \frac{5}{x-2} = \frac{x^2-6x-13}{x^2-x-2}$$

$$\text{Factor } x^2-x-2 \\ = (x-2)(x+1) \quad \begin{array}{r} -2 \\ -2 \\ -1 \end{array}$$

$$\text{LCD} = (x-2)(x+1)$$

$$\frac{2}{\cancel{x+1}} \cdot \frac{(x-2)}{(x-2)} - \frac{5}{\cancel{x-2}} \cdot \frac{(x+1)}{(x+1)} = \frac{(x^2-6x-13)}{(\cancel{x+1})(\cancel{x-2})}$$

$$2(x-2) - 5(x+1) = x^2-6x-13$$

$$2x-4-5x-5 = x^2-6x-13$$

$$-3x-9 = x^2-6x-13$$

$$0 = x^2-3x-4$$

$$0 = (x-4)(x+1)$$

$$x-4=0$$

$$x=4$$

$$x+1=0$$

$$x=-1$$

Verify $x=4$

$$\frac{2}{4+1} - \frac{5}{4-2} = \frac{4^2-6(4)-13}{4^2-4-2}$$

$$\frac{2}{5} - \frac{5}{2} = \frac{-21}{10} \checkmark$$

Verify $x=-1$

$$\frac{2}{-1+1} - \frac{5}{-1-2} =$$

REJECT $x=-1$

ugly stuff
we don't
need to check
because already
have $\div 0$

Mult all by LCD
cancel denoms
Rewrite remaining

dist $2(x-2)$
dist $-5(x+1)$

Notice $x^2 \Rightarrow$ strategy
is set=0, factor

combine like terms
add $3x, 9$ both sides

$$\text{factor } x^2-3x-4 \\ = (x-4)(x+1) \quad \begin{array}{r} -4 \\ -4 \\ -3 \end{array}$$

$$\frac{2}{x+1} \text{ undefined } x=-1$$

REJECT $x=-1$

$$\frac{5}{x-2} \text{ undefined } x=2$$

$$\frac{x^2-6x-13}{x^2-x-2} \text{ undefined } x=-1 \text{ and } x=2$$

$$\boxed{x=4} \text{ or } \boxed{\{4\}}$$

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$$(10) \quad \frac{-12}{x^2-9} = \frac{2}{x+3} - \frac{5}{x-3}$$

$$\text{Factor } x^2-9 = (x+3)(x-3)$$

$$\text{LCD} = (x+3)(x-3)$$

Multiply all by LCD

$$\frac{-12}{(x+3)(x-3)} \cdot \cancel{(x+3)(x-3)} = \frac{2}{\cancel{(x+3)}} \cdot \cancel{(x+3)}(x-3) - \frac{5 \cdot (x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$-12 = 2(x-3) - 5(x+3)$$

$$-12 = 2x - 6 - 5x - 15$$

$$-12 = -3x - 21$$

$$9 = -3x$$

$$-3 = x$$

Cancel denominators

Rewrite remaining terms

$$\text{dist } 2(x-3)$$

$$\text{dist } -5(x+3)$$

Combine like terms

add 21 both sides

divide -3 both sides

isolate x

Verify:

$$\left. \begin{array}{l} x^2-9 \\ x+3 \\ x-3 \end{array} \right\} \text{undefined } x=3, x=-3$$

Reject $x=-3$

NO SOLUTION

OR

$\emptyset$

OR

$\{\}$

Caution:

$\emptyset$ means "the empty set"

$\{\}$ means "the empty set"

Do NOT USE: $\{\emptyset\}$

("The set containing the empty set") when you mean NO SOLUTION

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⑪ Solve $\frac{a}{1+b} = c$ for b .

$$\frac{a}{\cancel{(1+b)}} \cdot \cancel{(1+b)} = c \cdot (1+b)$$

$$a = c(1+b)$$

$$a = c + bc$$

$$\frac{a-c}{c} = \frac{bc}{c}$$

$$\boxed{b = \frac{a-c}{c}}$$

$$\text{LCD} = 1+b$$

mult all terms by LCD.
cancel denoms.

$$\text{dist } c(1+b)$$

subtract c both sides
divide c both sides

⑫ Solve $\frac{1}{x} = \frac{1}{y} + \frac{1}{z}$ for y .

$$\text{LCD} = xyz$$

$$\frac{1}{\cancel{x}} \cdot \cancel{xyz} = \frac{1}{\cancel{y}} \cdot \cancel{xyz} + \frac{1}{\cancel{z}} \cdot \cancel{xyz}$$

mult all by LCD
cancel denoms

$$yz = xz + xy$$

Notice: y appears
in two locations!

Strategy: Collect y -terms
on same side and
factor out y .

$$yz - xy = xz$$

$$y(z-x) = xz$$

$$\boxed{y = \frac{xz}{z-x}}$$

subtract xy both sides

factor out y

divide both sides by $(z-x)$.

⑬ Solve $x = \frac{ab}{a+b}$ for a .

$$x(a+b) = ab$$

$$xa + xb = ab$$

mult by $\text{LCD} = (a+b)$

dist $x(a+b)$

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$$xb = ab - xa$$

subtract xa both sides
to collect all terms
containing a

$$xb = a(b-x)$$

$$\boxed{\frac{xb}{b-x} = a}$$

factor out a

divide by $(b-x)$.

(14) Substitute $\bar{C} = 240$ into given equation:

$$\bar{C} = \frac{x^2 + 75x + 5000}{x}$$

$$240 = \frac{x^2 + 75x + 5000}{x}$$

Multiply both sides
by $LCD = x$

$$240x = x^2 + 75x + 5000$$

$$0 = x^2 - 165x + 5000$$

Notice x^2 : Strategy is
set $= 0$ and factor

Factor $x^2 - 165x + 5000$

$$(x-40)(x-125) = 0$$

$$x-40 = 0 \quad x-125 = 0$$

$$x = 40 \quad x = 125$$

$$(x-40)(x-125) \quad \begin{array}{r} 5000 \\ -40 \quad -125 \\ \hline -165 \end{array}$$

-1, -5000
-2, -2500
-4, -1250
-5, -1000
-8, 625
-10, -500
-20, -250
-25, -200
-40, -125

Verify $\frac{x^2 + 75x + 5000}{x}$ is undefined
for $x = 0$ only.

$$\boxed{x = 40 \text{ bicycles or } x = 125 \text{ bicycles}}$$

(15) Solve $\frac{4}{x-3} - \frac{3}{x-2} = \frac{2x+1}{x^2-5x+6}$

Factor $x^2 - 5x + 6 = (x-2)(x-3)$ $\begin{array}{r} 6 \\ -2 \quad -3 \\ \hline -5 \end{array}$

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$$\frac{4}{(x-3)} \cdot (x-2)(x-3) - \frac{3}{(x-2)} \cdot (x-2)(x-3) = \frac{(2x+1)}{(x-2)(x-3)} \cdot (x-2)(x-3)$$

mult all by LCD
add ()
cancel denoms
rewrite
dist $4(x-2)$
 $-3(x-3)$

$$4(x-2) - 3(x-3) = 2x+1$$

$$4x-8 - 3x+9 = 2x+1$$

$$x+1 = 2x+1$$

$$0 = x$$

$$\frac{4}{x-3}, \frac{3}{x-2}, \frac{2x+1}{x^2-5x+6}$$

undefined for $x=3, x=2$.

$$\boxed{x=0} \text{ or } \boxed{\{0\}}$$

$$(16) \quad \frac{6}{t} - \frac{2}{t-1} = \frac{2-4t}{t^2-t}$$

$$6(t-1) - 2t = 2-4t$$

$$6t-6-2t = 2-4t$$

$$4t-6 = 2-4t$$

$$8t = 8$$

$$t = 1$$

$$\frac{6}{t}, \frac{2}{t-1}, \frac{2-4t}{t^2-t} \text{ undefined for } t=0 \text{ and } t=1$$

Reject $t=1$

$$\boxed{\text{No SOLUTION}} \text{ or } \boxed{\emptyset} \text{ or } \boxed{\{ \}}$$

$$\text{Factor } t^2-t = t(t-1)$$

$$\text{LCD} = t(t-1)$$

{ mult by LCD
cancel, rewrite }

Some steps done in head!

$$\text{dist } 6(t-1)$$

combine

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$$(17) \frac{5}{x+1} - \frac{7}{2} = \frac{-5x-9}{2x+2}$$

$$2 \cdot \cancel{(x+1)} \cdot \frac{5}{\cancel{(x+1)}} - \cancel{2} \cdot (x+1) \cdot \frac{7}{\cancel{2}} = \frac{\cancel{2} \cdot \cancel{(x+1)} \cdot (-5x-9)}{\cancel{2} \cdot \cancel{(x+1)}}$$

Factor $2 \cdot x + 2 = 2(x+1)$
LCD = $2(x+1)$

Add parentheses
Mult all terms by LCD
Cancel denominators

$$2 \cdot 5 - (x+1) \cdot 7 = (-5x-9)$$

$$10 - 7(x+1) = -5x-9$$

$$10 - 7x - 7 = -5x - 9$$

$$-7x + 3 = -5x - 9$$

$$-2x = -12$$

$$x = 6$$

rewrite remaining terms

simplify $2 \cdot 5 = 10$
move 7 to left side
of ().

dist $-7(x+1)$

combine $10-7$

subtract 3 both sides
add $5x$ both sides
isolate x

Verify:

$$\frac{5}{6+1} - \frac{7}{2} = \frac{-5(6)-9}{2(6)+2}$$

$$\frac{5}{7} - \frac{7}{2} = \frac{-39}{14} \checkmark$$

$$\frac{5}{x+1} \Rightarrow x \neq -1$$

$$\frac{-5x-9}{2x+2} \Rightarrow x \neq -1$$

we didn't get $x = -1$.

$$\boxed{x=6} \text{ or } \boxed{\{6\}}$$

$$(18) \text{ Simplify. } 5 - \frac{n^2-3n-4}{n^2-4}$$

subtract, need LCD.

$$= \frac{5(n^2-4) - (n^2-3n-4)}{n^2-4}$$

write w/ LCD,
subtract

$$= \frac{5n^2 - 20 - n^2 + 3n + 4}{n^2-4}$$

dist 5
& neg

COMMON QUESTION:
CAN WE CLEAR
the fraction?

No.

This is an expression,
not an equation. We
can only multiply by 1.
(not both sides -- there's
only one "side")

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$$= \frac{4n^2 + 3n - 16}{n^2 - 4}$$

$$= \boxed{\frac{4n^2 + 3n - 16}{(n+2)(n-2)}}$$

Factor $4n^2 + 3n - 16$

$$\frac{-64}{3}$$

does not factor

- 1, 64
- 2, 32
- 4, 16
- 8, 8

After chap 10:

$$\begin{aligned} b^2 - 4ac &= 9 - 4(4)(-16) \\ &= 9 + 256 = 265 \\ &\text{not a perfect square} \\ &\text{does not factor} \end{aligned}$$